JOURNAL OF ALGEBRA 91, 1–17 (1984)

Codimensions of T-Ideals and Hilbert Series of Relatively Free Algebras

VESSELIN DRENSKY

Institute of Mathematics,
Bulgarian Academy of Sciences, 1090 Sofia, P. O. Box 373, Bulgaria

Communicated by P. M. Cohn

Received March 31, 1982

INTRODUCTION

The main object of this paper is to study the T-ideal $\mathscr{M}_2$ of the identities satisfied by the algebra M_2 of 2×2 matrices over a field of characteristic 0. More precisely, the relatively free algebra of the variety of associative algebras var M_2 generated by M_2 is investigated. This algebra is isomorphic to the algebra of 2×2 generic matrices.

Procesi [12, p. 185] sets the problem of computing the dimensions of the homogeneous components of the identities in m variables satisfied by the algebra M_n of $n \times n$ matrices. Until now an answer has been obtained only in the case of the identities in two variables for M_2, by Formanek *et al.* [9]. Furthermore, particular results have been established about the cocharacters of $\mathscr{M}_2$ by Regev [16] and Berele [4]. For a background of the results concerning the identities of the matrix algebras cf the survey of Formanek [8].

In Section 3 of the present paper the Hilbert series (or Poincaré series) of the algebra generated by an arbitrary number of 2×2 generic matrices is computed. This is equivalent to the problem of Procesi in the case of matrices of second order. The cocharacters of the T-ideal $\mathscr{M}_2$ are also computed. In particular, new proofs of the theorems of Formanek *et al.* [9] and Berele [4, Theorem 3.5] are given. Codimensions of the proper subvarieties of var M_2 are also evaluated.

There is a relationship between the varieties of associative and Lie algebras generated by M_2 and $sl(2, K)$, respectively. In Section 4 the results obtained are transferred to the case of Lie algebras. Moreover, the theorem of Bahturin [1], which is an analog of the result of Formanek *et al.* [9], is generalized.

In order to obtain these results, it is useful to consider a more general situation of an arbitrary variety of associative algebras with 1 over a field of

0021-8693/84 $3.00

characteristic 0. This is done in Section 2. It turns out that the structure of the relatively free algebras (in a module-theoretic language) is completely determined by the proper multilinear polynomials which are not identities for the variety.

Another application of this general setup is given in Section 5, where the T-ideals generated by a commutator, respectively by a non-matrix identity of degree four, are studied.

The main results obtained in this paper are announced without proofs in [5].

1. Definitions and Notations

All associative algebras will be unitary and over a fixed field K of characteristic zero. The necessary definitions are given in [6] and [7]. We use notations similar to those of [6]. For any variety $\mathfrak{M}$ of associative algebras $F_m(\mathfrak{M})$ (resp. $F_m^{(n)}(\mathfrak{M})$, $F^{(n_1,\ldots,n_m)}(\mathfrak{M})$) is the relatively free algebra of rank m in $\mathfrak{M}$ with free generators $x_1,\ldots,x_m$ (resp. its homogeneous components of total degree n, or of degree n_i in x_i $(1 \leqslant i \leqslant m)$). The set of all multilinear polynomials in $F_n^{(n)}(\mathfrak{M})$ is denoted by $P_n(\mathfrak{M})$.

The set $P_n(\mathfrak{M})$ has a structure of a left S_n-module and $F_m^{(n)}(\mathfrak{M})$ has a structure of a left $GL(m,K)$-module [7] (S_n denotes the symmetric group). Let

$$b_{n_1,\ldots,n_m} = b_{n_1,\ldots,n_m}(\mathfrak{M}) = \dim F^{(n_1,\ldots,n_m)}(\mathfrak{M}),$$

$$b_m^{(n)} = b_m^{(n)}(\mathfrak{M}) = \dim F_m^{(n)}(\mathfrak{M}), \qquad c_n = c_n(\mathfrak{M}) = \dim P_n(\mathfrak{M}).$$

The sequence of numbers $c_1, c_2,\ldots$ is called the codimention sequence and the sequence of S_n-characters $\chi(P_1(\mathfrak{M})), \chi(P_2(\mathfrak{M})),\ldots$ is called the cocharacter sequence of the T-ideal $T(\mathfrak{M})$, determining $\mathfrak{M}$. The formal power series

$$H(\mathfrak{M}, t_1,\ldots, t_m) = \sum_{n_i \geqslant 0} b_{n_1,\ldots,n_m} t_1^{n_1} \cdots t_m^{n_m},$$

$$H_m(\mathfrak{M}, t) \sum_{n \geqslant 0} b_m^{(n)} t^n = H(\mathfrak{M}, t,\ldots, t)$$

are called Hilbert (or Poincaré) series of the relatively free algebra $F_m(\mathfrak{M})$. It is convenient to consider also the codimension series

$$c(\mathfrak{M}, t) = \sum_{n \geqslant 0} c_n t^n.$$

The knowledge of the series $H(\mathfrak{M}, t_1,\ldots, t_m)$ is equivalent to that of the Hilbert series $H(T(\mathfrak{M}), t_1,\ldots, t_m)$ of $T(\mathfrak{M}) \cap A_m$, where A_m is the free associative algebra of rank m.

Let $B_m^{(n)}(\mathfrak{M})$ be the linear subspace of $F_m^{(n)}(\mathfrak{M})$ spanned by the commutator products $[x_{i_1},...] \cdots [..., x_{i_n}]$, $B_m(\mathfrak{M}) = \sum_{n\geqslant 0} B_m^{(n)}(\mathfrak{M})$. We set $\Gamma_n(\mathfrak{M}) = B_n^{(n)}(\mathfrak{M}) \cap P_n(\mathfrak{M})$. The elements of $\Gamma_n(\mathfrak{M})$ are called proper multilinear polynomials of degree n. It is known that any subvariety of $\mathfrak{M}$ is determined by its identities in $\Gamma_n(\mathfrak{M})$, $n = 2, 3,...$. The linear spaces $\Gamma_n(\mathfrak{M})$ and $B_m^{(n)}(\mathfrak{M})$ are submodules of the S_n-module $P_n(\mathfrak{M})$ and of the $GL(m, K)$-module $F_m^{(n)}(\mathfrak{M})$, respectively. Let

$$\beta(\mathfrak{M}, t_1,..., t_m) = \sum_{n_i\geqslant 0} \beta_{n_1,...,n_m} t_1^{n_1} \cdots t_m^{n_m}$$

and

$$\beta_m(\mathfrak{M}, t) = \sum_{n\geqslant 0} \beta_m^{(n)} t^n = \beta(\mathfrak{M}, t,..., t)$$

be the Hilbert series of the graded spaces $B_m(\mathfrak{M})$, where

$$\beta_{n_1,...,n_m} = \beta_{n_1,...,n_m}(\mathfrak{M}) = \dim(B_m^{(n)}(\mathfrak{M}) \cap F^{(n_1,...,n_m)}(\mathfrak{M})),$$
$$\beta_m^{(n)} = \beta_m^{(n)}(\mathfrak{M}) = \dim B_m^{(n)}(\mathfrak{M}).$$

We shall also consider the series

$$\gamma(\mathfrak{M}, t) = \sum_{n\geqslant 0} \gamma_n t^n, \qquad \text{where} \quad \gamma_n = \gamma_n(\mathfrak{M}) = \dim \Gamma_n(\mathfrak{M}).$$

Here we set $\beta_m^{(0)} = \gamma_0 = 1$, $\beta_m^{(1)} = \gamma_1 = 0$.

Irreducible S_n- and $GL(m, K)$-modules are described by Young diagrams. We use the notation $M(p_1,..., p_r)$ (resp. $N_m(p_1,..., p_r)$ and $\chi(p_1,..., p_r)$) for the irreducible S_n-module (resp. the $GL(m, K)$-module and the S_n-character), corresponding to the partition $(p_1,..., p_r)$ of n. Formulas of the dimensions of $M(p_1,..., p_r)$ and $N_m(p_1,..., p_r)$ are given in [10, p. 81] and [18, p. 201]. We recall that

$$\dim N_m(p_1,..., p_m) = \prod_{m\geqslant i>j\geqslant 1} (p_j - p_i + i - j)(i-j)^{-1}$$
$$(p_1 \geqslant \cdots \geqslant p_m \geqslant 0, p_1 + \cdots + p_m = n). \tag{1}$$

The commutative polynomial algebra is denoted by $K[x_1,..., x_m]$. Its homogeneous components are irreducible $GL(m, K)$-modules, isomorphic to $N_m(n, 0,..., 0)$. The Hilbert series of this algebra is given by

$$G(\mathfrak{A}, t_1,..., t_m) = \prod_{i=1}^{m} (1 - t_i)^{-1}.$$

In the sequel we shall use without explicit reference the following identity of formal power series

$$\sum_{n\geq 0} \binom{n+a}{a} t^n = (1-t)^{-1-a}.$$

2. The Module Structure of Relatively Free Algebras

In this section $\mathfrak{M}$ is a proper subvariety of the variety of all associative algebras. The following two propositions give some restrictions on the parameters introduced above:

Proposition 2.1. *The codimension series $c(\mathfrak{M}, t)$ has a non-zero radius of convergence.*

Proof. By the theorem of Regev [15] there exists a positive real number λ such that $c_n(\mathfrak{M}) \leqslant \lambda^n$. This immediately implies the proposition because the series $c(\mathfrak{M}, t)$ is majorized by the series $1 + \lambda t + \lambda^2 t^2 + \cdots$ which has a non-zero radius of convergence.

The following statement has been proved independently by Berele [4, Corollary 4.12].

Proposition 2.2. *The coefficients $b_m^{(n)}$ of the Hilbert series $H_m(\mathfrak{M}, t)$ have a polynomial growth, i.e., there exist polynomials $f_m(n)$, $m = 1, 2,\ldots$, such that $b_m^{(n)} \leqslant f_m(n)$.*

Proof. By the Height Theorem of Shirshov [20, p. 128] there exist positive integers N and d such that the elements of $F_m(\mathfrak{M})$ are linear combinations of the terms $Y_{i_1}^{a_1} \cdots Y_{i_M}^{a_M}$, where $a_i \geqslant 0$, $M \leqslant N$ and Y_i are monomials in $F_m(\mathfrak{M})$ of degree not exceeding d. Therefore, the graded linear space $F_m(\mathfrak{M})$ is a homomorphic image of the graded space $\sum K[Y_{i_1},\ldots, Y_{i_N}]$. Here $Y_{i_1},\ldots, Y_{i_N}$ range over the set of all monomials in $x_1,\ldots, x_m$ of degree not more than d and the degree of Y_i in $K[Y_{i_1},\ldots, Y_{i_N}]$ equals the length of Y_i considered as a word in $x_1,\ldots, x_m$. Obviously, the series $H_m(\mathfrak{M}, t)$ is majorized by a finite sum of the Hilbert series of the polynomial algebras $K[Y_{i_1},\ldots, Y_{i_N}]$. But the coefficients of these series have polynomial growth.

We shall estalish that the module structure of the relatively free algebra of $\mathfrak{M}$ is completely determined by the structure of the S_n-modules $\Gamma_n(\mathfrak{M})$, $n = 2, 3,\ldots$.

Lemma 2.3. *Let $\Gamma_n(\mathfrak{M}) = \oplus M_D$, M_D being the irreducible S_n-modules corresponding to the Young diagrams $D \in J$. Then $B_m^{(n)} = \oplus N_D$, where N_D are irreducible $GL(m, K)$-modules corresponding to D and the summation is taken over all diagrams $D \in J$ with not more than m rows.*

Proof. Let $f(x_1,\dots,x_m)$ be a polynomial from $B_m^{(n)}(\mathfrak{M})$, homogeneous in each of $x_1,\dots,x_m$ and generating an irreducible $GL(m,K)$-submodule of $B_m^{(n)}(\mathfrak{M})$. It is clear that the linearization of $f(x_1,\dots,x_m)$ lies in $\Gamma_n(\mathfrak{M})$. Conversely, the symmetrization [7, Sect. 1] of every irreducible S_n-submodule of $\Gamma_n(\mathfrak{M})$ is in $B_m^{(n)}(\mathfrak{M})$. The assertion now follows immediately from [7, 1.3 and 1.5].

LEMMA 2.4. *Let $\{g_k^{(s)}(x_1,\dots,x_m) \mid k=1,\dots,\beta_m^{(s)}\}$ be a basis of $B_m^{(s)}(\mathfrak{M})$, let Q_s be the subspace of $F_m^{(n)}(\mathfrak{M})$ spanned by the polynomials $x_{i_1}\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m)$, $k=1,\dots,\beta_m^{(s)}$, and let $R_s=\sum_{t\geqslant s}Q_t$. Then*

(i) $R_0=F_m^{(n)}(\mathfrak{M})$;

(ii) *for every permutation $j_1,\dots,j_{n-s}$ of the integers $i_1,\dots,i_{n-s}$*

$$x_{j_1}\cdots x_{j_{n-s}}g_k^{(s)}(x_1,\dots,x_m)\equiv x_{i_1}\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m) \pmod{R_{s-1}};$$

(iii) *the set*

$$U_s=\{x_1^{a_1}\cdots x_m^{a_m}g_k^{(s)}(x_1,\dots,x_m) \mid a_1+\cdots+a_m = n-s,\ k=1,\dots,\beta_m^{(s)}\}$$

forms a basis of R_s modulo R_{s-1}.

Proof. (i) We shall use an idea of Volichenko [19, Theorem 1]. The free associative algebra A_m is a universal enveloping algebra of the free Lie algebra L_m. Let $l_1<l_2<\cdots$ be an ordered basis of L_m consisting of commutator monomials $[x_{i_1},\dots,x_{i_k}]$ such that $\deg l_1\leqslant \deg l_2\leqslant\cdots$. By the Poincaré–Birkhoff–Witt theorem A_m has a basis $l_1^{a_1}\cdots l_s^{a_s}$, $a_i\geqslant 0$. There is a homomorphism of A_m onto $F_m(\mathfrak{M})$ extending the map $x_i\to x_i$, $i=1,\dots,m$. Consequently, $F_m^{(n)}(\mathfrak{M})$ is spanned by the polynomials

$$x_{i_1}\cdots x_{i_{n-s}}[x_{l_1},\dots]\cdots[\dots,x_{l_s}],\qquad i_1\leqslant\cdots\leqslant i_{n-s},\quad s=0,2,3,\dots,n.$$

The elements $[x_{l_1},\dots]\cdots[\dots,x_{l_s}]$ can be linearly expressed in terms of $g_k^{(s)}(x_1,\dots,x_m)$, $k=1,\dots,\beta_m^{(s)}$, i.e. $F_m^{(n)}(\mathfrak{M})=R_0$.

(ii)
$$\begin{aligned} x_{i_1}\cdots x_{i_r}x_{i_{r+1}}\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m) &= x_{i_1}\cdots x_{i_{r-1}}x_{i_r}\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m)\\ &\quad + x_{i_1}\cdots[x_{i_r},x_{i_{r-1}}]\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m).\end{aligned}$$

Using the identity $ux=xu+[u,x]$ we shift to the right the commutator $[x_{i_r}, x_{i_{r-1}}]$ from the last summand, i.e., we express $x_{i_1}\cdots[x_{i_r},x_{i_{r-1}}]\cdots x_{i_{n-s}}g_k^{(s)}(x_1,\dots,x_m)$ as a sum of $x_{p_1}\cdots x_{p_{n-t}}g_l^{(t)}(x_1,\dots,x_m)$. In this expression all indices t are bigger than s. Hence, modulo R_{s+1}, we can exchange the positions of x_{i_r} and $x_{i_{r+1}}$ in $x_{i_1} \cdots x_{i_{n-s}} g_k^{(s)}(x_1,..., x_m)$. This proves (ii).

(iii) By (ii) it is sufficient to establish that the set $U = U_0 \cup U_2 \cup U_3 \cup \cdots \cup U_n$ consists of linearly independent elements. The polynomials $g_k^{(s)}(x_1,..., x_m)$ are expressed by commutator products. Therefore $g_k^{(s)}(1 + x_1, x_2,..., x_m) = g_k^{(s)}(x_1, x_2,..., x_m)$. Let

$$\sum \mu_{\alpha k s} x_1^{\alpha_1} \cdots x_m^{\alpha_m} g_k^{(s)}(x_1,..., x_m) = 0,$$

$$\alpha_1 + \cdots \alpha_m + s = n, k = 1,..., \beta_m^{(s)}$$

be a linear relation between polynomials from U. We replace x_i by $1 + x_i$, $i = 1,..., m$, and take the components homogeneous in each x_i, $i = 1,..., m$. Let s_0 be the least integer such that there exists a $\mu_{\alpha k s_0}$ different from zero. We obtain

$$\sum_{l=1}^{\beta_m^{(s_0)}} \mu_{\alpha l s_0} g_l^{(s_0)}(x_1,..., x_m) = 0.$$

Hence, all the coefficients $\mu_{\alpha k s}$ are zero.

COROLLARY 2.5. (i) $c_n(\mathfrak{M}) = \sum_{s=0}^{n} \binom{n}{s} \gamma_s(\mathfrak{M})$;

(ii) $c(\mathfrak{M}, t) = (1 - t)^{-1} \gamma(\mathfrak{M}, t(1 - t)^{-1})$.

Proof. Let $f_k^{(s)}(x_1,..., x_s)$, $k = 1,..., \gamma_s$, be a basis of the linear space $\Gamma_s(\mathfrak{M})$. By Lemma 2.4, the set

$$\{x_{i_1} \cdots x_{i_{n-s}} f_k^{(s)}(x_{j_1},..., x_{j_s}), i_1 < \cdots < i_{n-s}, j_1 < \cdots < j_s,$$
$$\{i_1,..., i_{n-s}, j_1,..., j_s\} = \{1,..., n\}, k = 1,..., \gamma_s, s = 0, 2, 3,..., n\}$$

forms a basis of $P_n(\mathfrak{M})$. Part (i) of the corollary follows immediately after counting the elements of this basis, and (ii) is another way to express (i), since

$$c(\mathfrak{M}, t) = \sum_{n \geqslant 0} c_n t^n = \sum_{n \geqslant 0} \sum_{s=0}^{n} \binom{n}{s} \gamma_s t^n = \sum_{s \geqslant 0} \gamma_s \left(\sum_{n \geqslant s} \binom{n}{s} t^n \right)$$
$$= \sum_{s \geqslant 0} \gamma_s t^s (1 - t)^{-s-1} = (1 - t)^{-1} \gamma(\mathfrak{M}, t(1 - t)^{-1}).$$

THEOREM 2.6. *For any variety $\mathfrak{M}$ of associative algebras the $GL(m, K)$-modules $F_m(\mathfrak{M})$ and $K[x_1,..., x_m] \otimes_K B_m(\mathfrak{M})$ are isomorphic.*

Proof. In the notation of Lemma 2.4 the spaces R_s are $GL(m, K)$-modules. Each factor module R_s/R_{s+1} is isomorphic to the tensor product $K[x_1, \dots, x_m]^{(n-s)} \otimes B_m^{(s)}(\mathfrak{M})$, $s = 0, 2, 3, \dots, n$. The complete reducibility of $GL(m, K)$-modules implies that the modules $F_m^{(n)}(\mathfrak{M})$ and $\bigoplus_{s=0}^{n} (K[x_1, \dots, x_m]^{(n-s)} \otimes B_m^{(s)}(\mathfrak{M}))$ are isomorphic, which proves the theorem.

COROLLARY 2.7. (i) $H(\mathfrak{M}, t_1, \dots, t_m) = \prod_{i=1}^{m} (1 - t_i)^{-1} \beta(\mathfrak{M}, t_1, \dots, t_m)$;

(ii) $H_m(\mathfrak{M}, t) = (1 - t)^{-m} \beta_m(\mathfrak{M}, t)$;

(iii) *Let $k(p_1, \dots, p_m)$ and $k_1(p_1, \dots, p_m)$ denote the number of copies of $N_m(p_1, \dots, p_m)$ in $F_m(\mathfrak{M})$ and $B_m(\mathfrak{M})$, respectively. Then $k(p_1, \dots, p_m) = \sum k_1(q_1, \dots, q_m)$, where the summation is over all the partitions $(q_1, \dots, q_m)$ such that $p_1 \geqslant q_1 \geqslant p_2 \geqslant q_2 \geqslant \cdots \geqslant p_m \geqslant q_m$.*

Proof. Parts (i) and (ii) express that the Hilbert series of the tensor product of the two modules $K[x_1, \dots, x_m]$ and $B_m(\mathfrak{M})$ equals the product of the series of the factors.

(iii) By the rule for the decomposition of a tensor product of $GL(m, K)$-modules [2, Chap. 8, Sect. 8]

$$K[x_1, \dots, x_m]^{(s)} \otimes N_m(p_1, \dots, p_m)$$
$$\cong \bigoplus N_m(p_1 + a_1, p_2 + a_2, \dots, p_m + a_m),$$

where $a_1 + \cdots + a_m = s$, $p_2 + a_2 \leqslant p_1$, $p_3 + a_3 \leqslant p_2, \dots, p_m + a_m \leqslant p_{m-1}$ and this is (iii).

Remark 2.8. As follows from [7, Propositions 1.3 and 1.5] and [4, Theorem 2.7] the description of the cocharacter sequence is equivalent to the knowledge of the number of copies of $N_m(p_1, \dots, p_m)$ in $F_m^{(n)}(\mathfrak{M})$. Hence, Lemma 2.3 and Corollary 2.7 reduce the computation of the cocharacter sequence of $T(\mathfrak{M})$ to that of the S_n-characters of $\Gamma_n(\mathfrak{M})$, $n = 0, 2, 3, \dots$.

3. THE MATRIX VARIETY OF SECOND ORDER

Let $\mathfrak{W} = \operatorname{var} M_2$ be the variety of associative algebras generated by the 2×2 matrix algebra. Berele [4, Theorem 3.5] has determined the quantities of the characters $\chi(p_1, p_2)$ in the cocharacter sequence of $\mathscr{M}_2$. The following theorem solves the problem completely.

THEOREM 3.1. $\chi(P_n(\operatorname{var} M_2)) = \sum k(p_1, p_2, p_3, p_4)\, \chi(p_1, p_2, p_3, p_4)$, *where $p_1 + p_2 + p_3 + p_4 = n$, $p_1 \geqslant p_2 \geqslant p_3 \geqslant p_4$ and*

(i) $k(n)=1$;

(ii) $k(p_1,p_2)=(p_1-p_2+1)p_2$, *if* $p_2>0$;

(iii) $k(p_1,1,1,p_4)=(p_1+1)(2-p_4)-1$;

(iv) $k(p_1,p_2,p_3,p_4)=(p_1-p_2+1)(p_2-p_3+1)(p_3-p_4+1)$ *in all other cases.*

Proof. The case (i) is trivial.

If $n>1$ then it follows from [7, Sect. 4] that $B_m^{(n)}(\mathfrak{W})$ is decomposed in a sum of non-isomorphic irreducible $GL(m,K)$-modules $N_m(q_1,q_2,q_3)$, where $q_1+q_2+q_3=n$, $q_1\geqslant q_2\geqslant q_3$, $q_2>0$, and the equalities $q_1=q_2=1$ imply $n=2$.

Let $p_2>0$. By Corollary 2.7 it suffices to count these triplets (q_1,q_2,q_3) for which $p_1\geqslant q_1\geqslant p_2\geqslant q_2\geqslant p_3\geqslant q_3\geqslant p_4$. There are p_1-p_2+1 possibilities for q_1, p_2-p_3+1 for q_2 and p_3-p_4+1 for q_3. If $p_3=0$ we have to subtract from all $(p_1-p_2+1)(p_2-p_3+1)(p_3-p_4+1)$ cases those for which $q_2=0$ (p_1-p_2+1 cases) and if $p_2=p_3=1$ then we have to subtract the cases of $q_1=q_2=q_3=1$.

THEOREM 3.2. $H_m(\operatorname{var} M_2,t) = (1-t)^{-m}\ (\theta(t,t,t)-(1-t)^{-m}+1-\binom{m}{3}t^3)$, *where* $m\geqslant 3$,

$$\theta(x,y,z)=\frac{2}{(m-1)!\,(m-2)!\,(m-3)!\,x^2y}\,\frac{\partial^{3(m-3)}}{\partial x^{m-3}\partial y^{m-3}\partial z^{m-3}}\times\left(\frac{(xyz)^{m-3}\,x^2y(1-x^2y)}{(1-xyz)(1-xy)^3\,(1-x)^3}\right). \tag{2}$$

Proof. Obviously, $(1-t)^{-m}-1=\sum_{n>0}\dim N_m(n)\,t^n$, $\binom{m}{3}t^3=\dim N_m(1,1,1)\,t^3$. Using Corollary 2.7 and the description of $B_m(\mathfrak{W})$ [7, Sect. 4] it is sufficient to prove that

$$\sum \dim N_m(p_1,p_2,p_3)\,t^{p_1+p_2+p_3}=\theta(t,t,t),$$

where the summation is over all $p_1\geqslant p_2\geqslant p_3\geqslant 0$. It follows from (1) that

$$\dim N_m(p_1,p_2,p_3)=\binom{p_1+m-1}{m-3}\binom{p_2+m-2}{m-3}\binom{p_3+m-3}{m-3}(p_1-p_2+1)\times(p_2-p_3+1)(p_1-p_3+2)(m-1)^{-1}(m-2)^{-2}.$$

Hence $\sum \dim N_m(p_1, p_2, p_3)\, t^{p_1+p_2+p_3} = \theta(t, t, t)$, where

$$\theta(x, y, z) = (m-1)^{-1}\,(m-2)^{-2}\,x^{-2}y^{-1}\psi(x, y, z).$$

$$\psi(x, y, z) = \sum \binom{p_1+2+m-3}{m-3}\binom{p_2+1+m-3}{m-3}\binom{p_3+m-3}{m-3}$$
$$\times (p_1-p_2+1)(p_2-p_3+1)(p_1-p_3+2)$$
$$\times x^{p_1+2}y^{p_2+1}z^{p_3}.$$

The identities below are verified by direct computation:

$$\sum_{r,s,t} a_{rst}\binom{r+k}{k}\binom{s+l}{l}\binom{t+m}{m}x^r y^s z^t$$
$$= \frac{1}{k!\,l!\,m!}\frac{\partial^{k+l+m}}{\partial x^k\,\partial y^l\,\partial z^m}\left(x^k y^l z^m \sum_{r,s,t} a_{rst}x^r y^s z^t\right),$$
$$\times \sum_{q_i \geqslant 0} (q_1+1)(q_2+1)(q_1+q_2+2)\,u_1^{q_1}u_2^{q_2}$$
$$= \left(\sum (q_1+1)^2 u_1^{q_1}\right)\sum (q_2+1)\,u_2^{q_2}$$
$$+ \left(\sum (q_1+1)\,u_1^{q_1}\right)\sum (q_2+1)^2\,u_2^{q_2}$$
$$= (2(1-u_1)^{-3} - (1-u_1)^{-2})(1-u_2)^{-2}$$
$$+ (1-u_1)^{-2}\,(2(1-u_2)^{-3} - (1-u_2)^{-2})$$
$$= 2(1-u_1u_2)(1-u_1)^{-3}\,(1-u_2)^{-3}.$$

Therefore we establish

$$\psi(x, y, z) = \frac{1}{((m-3)!)^3}\frac{\partial^{3(m-3)}}{\partial x^{m-3}\,\partial y^{m-3}\,\partial z^{m-3}}\,(xyz)^3\,\varphi(x, y, z),$$

where

$$\varphi(x, y, z) = \sum (p_1-p_2+1)(p_2-p_3+1)(p_1-p_3+2)\,x^{p_1}y^{p_2}z^{p_3}$$
$$= 2(1-x^2y)(1-xyz)^{-1}\,(1-xy)^{-3}\,(1-x)^{-3}.$$

It gives the desired result.

The case $m = 2$ is settled by the theorem of Formanek *et al.* [9]. We give an alternate proof of their result, using the general approach developed above and in [7].

THEOREM 3.3 [9].

$$H(\operatorname{var} M_2, t_1, t_2) = (1-t_1)^{-1}(1-t_2)^{-1}(1+t_1t_2(1-t_1t_2)^{-1} \times (1-t_1)^{-1}(1-t_2)^{-1}).$$

Proof. In the view of Corollary 2.7, it suffices to compute $\beta(\mathfrak{W}, t_1, t_2)$. Consider the $GL(2, K)$-module $N_2(q+r, r)$. It is isomorphic to the submodule of the free associative algebra A_2 generated by $[x_1, x_2]^q x_1^r$. Its basis consists of homogeneous polynomials of degree n_i in x_i, $n_i \geqslant q$, $i = 1, 2$, $n_1 + n_2 = n$ (cf. [7, the proof of Corollary 4.7]). Hence

$$\begin{aligned} H(N_2(q+r, q), t_1, t_2) &= (t_1t_2)^q (t_1^r + t_1^{r-1}t_2 + \cdots + t_2^r) \\ &= (t_1t_2)^q (t_1^{r+1} - t_2^{r+1})(t_1 - t_2)^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} \beta(\mathfrak{W}, t_1, t_2) &= 1 + \sum_{q>0, r\geqslant 0} H(N_2(q+r, q), t_1, t_2) \\ &= 1 + (t_1 - t_2)^{-1} \sum_{q>0} (t_1t_2)^q \sum_{r>0} (t_1^r - t_2^r) \\ &= 1 + t_1t_2(1 - t_1t_2)^{-1}. \end{aligned}$$

THEOREM 3.4. *For any proper subvariety $\mathfrak{U}$ of $\mathfrak{W}$ there exists an integer-valued polynomial $f(n)$ such that $c_n(\mathfrak{U}) \leqslant 2^n f(n)$.*

Proof. It follows from [14, Corollary 2] that any proper subvariety $\mathfrak{U}$ of $\mathfrak{W}$ satisfies the identity $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = 0$. On the other hand, the proof of Theorem 4.2 [7] exhibits the concrete form of the generator f_{pqr} of the irreducible S_n-submodule M_{pqr} of $\Gamma_n(\mathfrak{W})$, corresponding to the partition $(p+q+r, p+q, p)$. It follows directly from the expression for f_{pqr} that $f_{pqr} = 0$ is an identity for $\mathfrak{U}$ if $p + q > k$. Thus, only those M_{pqr} enter the decomposition of $\Gamma_n(\mathfrak{U})$, for which $p + q \leqslant k$, $3p + 2q + r = n$. Fixing p and q, one sees by the hook formula [10, p. 81] that the dimension of M_{pqr} is an integer-valued polynomial in r. Consequently, $\gamma_n = \dim \Gamma_n(\mathfrak{U})$ has polynomial growth. By Corollary 2.5,

$$c_n(\mathfrak{U}) = \sum_{s=0}^{n} \binom{n}{s} \gamma_s \leqslant \max_{s\leqslant n} \gamma_s \sum_{s=0}^{n} \binom{n}{s} = 2^n \max_{s\leqslant n} \gamma_s,$$

which proves the theorem.

Remark 3.5. Regev [16] has proved that $c_n(\mathfrak{W}) = \mathcal{O}(4^n n^{-3/2})$. Together with Theorem 3.4, this gives a numerical expression of the fact that the structure of the subvarieties of $\mathfrak{W}$ is simpler than that of $\mathfrak{W}$.

4. The Variety of Lie Algebras Generated by the Three-Dimensional Simple Algebra

Let $\mathfrak{V} = \operatorname{var} sl(2, K)$ be the variety of Lie algebras generated by the Lie algebra of 2×2 matrices of trace zero. By [7, Sect. 4], $F_m(\mathfrak{V})$ is decomposed in a sum of those non-isomorphic $GL(m, K)$-modules $N_m(p+q+r, p+q, p)$ for which $p+q>0$ and at least one of the integers q and r is odd. The Hilbert series of $F_m(\mathfrak{V})$ is computed in the following theorem:

THEOREM 4.1. *$H_m(\mathfrak{V}, t) = 4^{-1}(3\theta(t, t, t) - \theta(t, -t, -t) - \theta(-t, t, -t) - \theta(-t, -t, t)) + 2^{-1}((1+t)^{-m} - (1-t)^{-m}) + mt$, $m \geqslant 3$ ($\theta(x, y, z)$ is defined by (2)).*

Proof. It is a consequence of the description of $F_m(\mathfrak{V})$ mentioned above that

$$\begin{aligned} H_m(\mathfrak{V}, t) = & \sum_{p,q,r \geqslant 0} \dim N_m(p+q+r, p+q, p)\, t^{3p+2q+r} \\ & - \sum_{p,q,r \geqslant 0} \dim N_m(p+2q+2r, p+2q, p)\, t^{3p+4q+2r} \\ & - \sum_{r \geqslant 0} \dim N_m(2r+1)\, t^{2r+1} + \dim N_m(1)\, t. \end{aligned}$$

We shall compute the sums:

$$\begin{aligned} \theta(x, y, z) &= \sum a_{pqr} x^{p+q+r} y^{p+q} z^p, \\ a_{pqr} &= \dim N_m(p+q+r, p+q, p).\ 3\theta(x, y, z) \\ &\quad - \theta(-x, -y, z) - \theta(-x, y, -z) - \theta(x, -y, -z) \\ &= \sum a_{pqr}(3 - (-1)^r - (-1)^q - (-1)^{q+r})\, x^{p+q+r} y^{p+q} z^p. \end{aligned}$$

Note that

$$3 - (-1)^r - (-1)^q - (-1)^{q+r} = \begin{cases} 4, & \text{if at least one of } q \text{ and } r \text{ is odd}, \\ 0, & \text{if both } q \text{ and } r \text{ are even}. \end{cases}$$

Hence

$$\begin{aligned} & 4^{-1}(3\theta(t, t, t) - \theta(-t, -t, t) - \theta(-t, t, -t) - \theta(t, -t, -t)) \\ & = \sum_{p,q,r \geqslant 0} a_{pqr} t^{3p+2q+r} - \sum_{p,q,r \geqslant 0} a_{p\,2q\,2r} t^{3p+4q+2r}. \end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{r\geqslant 0} \dim N_m(2r+1)\, t^{2r+1} &= 2^{-1} \sum_{r\geqslant 0} \dim N_m(r)(t^r - (-t)^r) \\ &= 2^{-1}((1-t)^{-m} - (1+t)^{-m}), \qquad \dim N_m(1) = m.\end{aligned}$$

The proof of the theorem is complete.

In the case $m = 2$ we give a new proof of the theorem of Bahturin [1]:

THEOREM 4.2. $H(\mathfrak{B}, t_1, t_2) = t_1 t_2 (1 + t_1 + t_2)(1 - t_1^2)^{-1} (1 - t_2^2)^{-1} (1 - t_1 t_2)^{-1}$.

Proof. The modules $F_2(\mathfrak{B})$ and $\sum_{q>0,r\geqslant 0} N_2(q+r, q)/\sum_{q>0,r\geqslant 0} N_2(2q+2r, 2q)$ are isomorphic. Therefore

$$\begin{aligned}H(\mathfrak{B}, t_1, t_2) = H\left(\sum_{q>0,r\geqslant 0} N_2(q+r, q), t_1, t_2\right) \\ - H\left(\sum_{q>0,r\geqslant 0} N_2(2q+2r, 2q), t_1, t_2\right),\end{aligned}$$

$$\begin{aligned}H&\left(\sum_{q>0,r\geqslant 0} N_2(2q+2r, 2q), t_1, t_2\right) \\ &= \sum_{q>0,r\geqslant 0} (t_1 t_2)^{2q} (t_1^{2r+1} - t_2^{2r+1})(t_1 - t_2)^{-1} \\ &= (t_1 t_2)^2 (1 - (t_1 t_2)^2)^{-1} (t_1 - t_2)^{-1} (t_1(1 - t_1)^{-1} - t_2(1 - t_2^2)^{-1}) \\ &= (t_1 t_2)^2 (1 - t_1 t_2)^{-1} (1 - t_1^2)^{-1} (1 - t_2^2)^{-1}.\end{aligned}$$

Proving Theorem 3.3, we showed that

$$H\left(\sum_{q>0,r\geqslant 0} N_2(q+r, q), t_1, t_2\right) = t_1 t_2 (1 - t_1 t_2)^{-1} (1 - t_1)^{-1} (1 - t_2)^{-1}.$$

Consequently,

$$\begin{aligned}H(\mathfrak{B}, t_1, t_2) &= t_1 t_2 (1 - t_1 t_2)^{-1} (1 - t_1)^{-1} (1 - t_2)^{-1} \\ &\quad \times (1 - t_1 t_2 (1 + t_1)^{-1} (1 + t_2)^{-1})) \\ &= t_1 t_2 (1 + t_1 + t_2)(1 - t_1 t_2)^{-1} (1 - t_1)^{-1} (1 - t_2)^{-1}.\end{aligned}$$

The next proposition is similar to Theorem 3.4:

THEOREM 4.3. *The codimension sequence of any proper subvariety of $\mathfrak{B}$ has a polynomial growth.*

Proof. By the theorem of Razmyslov [13, Theorem 3] any subvariety $\mathfrak{U}$ of $\mathfrak{B}$ lies in $\mathfrak{N}_k\mathfrak{A}$. The bound on the codimensions of $\mathfrak{U}$ is obtained in the same way as in Theorem 3.4 (cf. also [3]).

5. NON-MATRIX VARIETIES

Let $\mathfrak{L}_d$ be the variety of associative algebras determined by the left normed commutator of length $d+1$

$$[x_1, x_2, \ldots, x_{d+1}] = 0. \tag{3}$$

LEMMA 5.1. *There exists a positive integer $e = e(d)$ such that $\Gamma_n(\mathfrak{L}_d)$ is generated as a linear space by the products*

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2s-1}}, x_{i_{2s}}]\, g(x_{i_{2s+1}}, \ldots, x_{i_n}), \tag{4}$$

where $i_1 < i_2 < \cdots < i_{2s}$ and $g(x_1, \ldots, x_{n-2s})$ is a polynomial of degree not higher than e.

Proof. It is well known that $\mathfrak{L}_d$ satisfies the identity

$$[x_1, x_2, x_3] \cdots [x_{3k-2}, x_{3k-1}, x_{3k}] = 0. \tag{5}$$

As a linear space, $\Gamma_n(\mathfrak{L}_d)$ is spanned by

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2r-1}}, x_{i_{2r}}][x_{i_{2r+1}}, \ldots] \cdots [\ldots, x_{i_n}],$$

where $[x_{i_{2r+1}}, \ldots], \ldots, [\ldots, x_{i_n}]$ are commutators of length larger than 2. By (3) and (5), the length and the quantity of such commutators are bounded by d and $k-1$, respectively. Hence $\Gamma_n(\mathfrak{L}_d)$ is spanned by

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2r-1}}, x_{i_{2r}}] f_1(x_{i_{2r+1}}, \ldots, x_{i_n}), \tag{6}$$

where $\deg f_1 \leqslant d(k-1)$. The polynomial

$$[y_1, y_2][y_3, y_4] + [y_1, y_3][y_2, y_4] \tag{7}$$

is a consequence of the commutator of length three $[y_1, y_2, y_3]$. It is easy to show that one can replace any of the commutators in (6) by polynomials which are specializations of (7). Using (7), one begins to order $x_{i_1}, \ldots, x_{i_{2r}}$ in (6). At every exchange of places of two commutators or of two elements lying in successive commutators, a commutator of length four or a specialization of (7) appears:

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] = [x_{i_3}, x_{i_4}][x_{i_1}, x_{i_2}] + [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]],$$

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] = -[x_{i_1}, x_{i_3}][x_{i_2}, x_{i_4}] + ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] + [x_{i_1}, x_{i_3}][x_{i_2}, x_{i_4}]).$$

In view of (5), beginning at the kth step of this ordering of $x_{i_1},..., x_{i_{2r}}$, one can change the places of the elements freely. Therefore, the polynomials (6) are linear combinations of

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2s-1}}, x_{i_{2s}}] f_2(x_{i_{2s+1}},..., x_{i_{2r}}) f_3(x_{i_{2r+1}},..., x_{i_n}),$$

where f_2 is a product of specializations of (7) and of commutators of length 3 and 4, f_3 is a product of commutators of length larger than two, $i_1 < i_2 < \cdots < i_{2s}$, $\deg f_2 \leqslant 4(k-1)$, $\deg f_3 \leqslant d(k-1)$. Consequently, one can set $e = (d+4)(k-1)$.

Corollary 5.2. *The dimensions of the S_n-modules $\Gamma_n(\mathfrak{L}_d)$ have polynomial growth in n.*

Proof. Let M_r, $r = 0, 1,..., e$, be the linear space of the polynomials $g(x_1,..., x_r)$ of degree r from (4). Obviously $\dim M_r \leqslant e!$ Hence $\dim \Gamma_n(\mathfrak{L}_d) \leqslant (\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{e})\, e!$, and this expression is a polynomial in n.

Let G_t be the Grassman algebra with 1 and with t generators, i.e., G_t is the exterior algebra of the space K^t, and let $G = G_\infty$.

Proposition 5.3. *The algebra*

$$R = G \otimes (\underbrace{G_t \otimes \cdots \otimes G_t}_{l})$$

belongs to $\mathfrak{L}_d$ for a suitable d.

Proof. Let $e_1, e_2,...$ and $u_{1r},..., u_{tr}$ be the generators of G and of the rth copy of the algebra G_t, respectively. In order to verify that $R \in \mathfrak{L}_d$, it suffices to substitute in (3)

$$a_k = \prod e_i^{\delta_i} \otimes \left(\bigotimes_{r=1}^{l} (u_{1r}^{\varepsilon_{1r}} \cdots u_{tr}^{\varepsilon_{tr}}) \right),$$

$$\delta_i, \varepsilon_{jr} = 0, 1, \qquad k = 1,,..., d+1,$$

instead of $x_1, x_2,..., x_{d+1}$. If $[a_1, a_2,..., a_{d+1}] \neq 0$, then the factor u_{jr} enters the expression of $a_1, a_2,..., a_{d+1}$ not more than once. Therefore, if the integer

d is large enough then there exist three consecutive elements a_m, a_{m-1}, a_{m-2} of the type $f_\alpha \otimes 1 \otimes \cdots \otimes 1, f_\alpha \in G$. But, for every $a \in R$,

$$[a, f_{m-1} \otimes 1 \otimes \cdots \otimes 1, f_{m-2} \otimes 1 \otimes \cdots \otimes 1] = 0, \qquad \text{i.e.,} \quad R \in \mathfrak{L}_d.$$

THEOREM 5.4. *There exist polynomials $p_d(n)$ and $q_d(n)$, such that*

$$2^n p_d(n) \leqslant c_n(\mathfrak{L}_d) \leqslant 2^n q_d(n) \qquad \text{and} \qquad \deg_{d\to\infty} p_d(n) \to \infty. \tag{8}$$

Proof. The right inequality (8) follows from Corollary 5.2 (cf. the proof of Theorem 3.4). By analogy, in order to prove the left inequality (8), it suffices to show that there exists a polynomial $h_d(n)$ such that $\dim \Gamma_{2n}(\mathfrak{L}_d) \geqslant h_d(2n)$ and $\deg_{d\to\infty} h_d(n) \to \infty$. The linearization of the polynomial

$$v(x_1, x_2, \ldots, x_{2n}) = s_{2n}(x_1, x_2, \ldots, x_{2n})[x_1, x_2]^{l-1}, \tag{9}$$

$n = 1, 2, \ldots$ (s_{2n} denoting the standard identity), generates an irreducible $S_{2n-2l-2}$-submodule M_n of $\Gamma_{2n+2l-2}(\mathfrak{L}_d)$, corresponding to the partition $(l^2, 1^{2n-2})$. By the hook formula, fixing l, $\dim M_n$ is a polynomial of degree $2l$ in n. In view of Proposition 5.3, the proof of the theorem will be completed if one shows that (9) is not an identity for the algebra

$$G \otimes \underbrace{G_2 \otimes \cdots \otimes G_2}_{l}.$$

It follows by direct verification that

$$\begin{aligned} &v(a_1, a_2, \ldots, a_{2n}) \\ &\quad = 2^l l!(2n-2)!\, n e_3 \cdots e_n \otimes u_{11} u_{21} \otimes \cdots \otimes u_{1l} u_{2l} \neq 0, \end{aligned}$$

where $a_1 = 1 \otimes u_{11} \otimes \cdots \otimes u_{1l}$, $a_2 = 1 \otimes u_{21} \otimes \cdots \otimes u_{2l}$, $a_3 = e_3 \otimes 1 \otimes \cdots \otimes 1, \ldots, a_{2n} = e_{2n} \otimes 1 \otimes \cdots \otimes 1$.

THEOREM 5.5. *There exists a polynomial $f(t)$ (depending on d and m), such that*

$$H_m(\mathfrak{L}_d, t) = f(t)(1-t)^{-m}$$

Proof. By Corollary 2.7, it is sufficient to prove that $\beta_m(\mathfrak{L}_d, t) = f(t)$ for a suitable polynomial $f(t)$, i.e., $B_m^{(n)}(\mathfrak{L}_d) = 0$ for n large enough. The conclusion of Lemma 5.1 remains true if in its hypothesis $\Gamma_n(\mathfrak{L}_d)$ is replaced by $B_m^{(n)}(\mathfrak{L}_d)$. Then, if $n > m + e$, two equal x_{i_a}, x_{i_b} appear in the product $[x_{i_1}, x_{i_2}] \cdots [x_{i_{2s-1}}, x_{i_{2s}}]$ from (4), i.e., all elements (4) are zero.

By virtue of Proposition 5.3, $G \otimes G_t \otimes \cdots \otimes G_t$ belongs to $\mathfrak{L}_d$.

Conjecture 5.6. The variety $\mathfrak{L}_d$ is generated by a finite set of algebras of the type $G \otimes G_{i_1} \otimes \cdots \otimes G_{i_s}$ and $G_{j_1} \otimes \cdots \otimes G_{j_t}$.

This has been verified when $d \leqslant 4$: $\mathfrak{L}_1 = \mathfrak{A} = \operatorname{var} G_1$; $\mathfrak{L}_2 = \operatorname{var} G$ [11]; $\mathfrak{L}_3 = \operatorname{var}(G, G_2 \otimes G_2)$, as is seen from the description of $\Gamma_n(\mathfrak{L}_3)$ [19]; $\mathfrak{L}_4 = \operatorname{var}(G \otimes G_2, G_2 \otimes G_2 \otimes G_2)$ [17].

The description of the varieties satisfying a non-matrix identity of the fourth degree is given in [6]. They all are subvarieties of the varieties $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$ determined, respectively, by the identities

$$[[x_1, x_2], [x_3, x_4]] = 0, \qquad [x_1, x_2]^2 = 0, \qquad [x_2, x_1, x_1, x_1] = 0.$$

Using the results of Section 2, we can describe the module structure of the relatively free algebras of these varieties. For instance:

PROPOSITION 5.7. (i) $c_n(\mathfrak{M}_1) = 2^{n-1}(n-1) + 1 - \binom{n}{2} + 2\binom{n}{4}$,

(ii) $c_n(\mathfrak{M}_2) = 2^{n-1}(n-1) + 1 - \binom{n}{2} + 3\binom{n}{4} + 4\binom{n}{5}$,

(iii) $c_n(\mathfrak{M}_3) = 2^{n-1} + 2\binom{n}{3} + 5\binom{n}{4} + 4\binom{n}{5}$, $n \geqslant 1$.

Proof. By virtue of [6, Proposition 3.4],

$$\Gamma_2(\mathfrak{M}_1) = M(1^2), \qquad \Gamma_3(\mathfrak{M}_1) = M(2, 1),$$

$$\Gamma_4(\mathfrak{M}_1) = M(3, 1) + M(2^2) + M(1^4), \qquad \Gamma_{2k-1}(\mathfrak{M}_1) = M(2k-2, 1),$$

$$\Gamma_{2k}(\mathfrak{M}_1) = M(2k-1, 1) + M(1^{2k}), \qquad k \geqslant 3.$$

Therefore, (i) follows from Corollary 2.5 and the hook formula. The proofs of (ii) and (iii) are analogous.

ACKNOWLEDGMENTS

Acknowledgments are due to L. Avramov for useful discussion and for help and attention during the work on this paper, and to G. Almkvist and E. Vinberg for discussions concerning, respectively, the theory of summation of series and the theory of representations of $GL(m, K)$.

REFERENCES

1. JU. A. BAHTURIN, 2-Variable identities of the Lie algebra $sl(2, k)$, *Trudy Sem. Petrovsk.* 5 (1979), 205–208 (Russian).
2. A. BARUT AND R. RĄCZKA, "Theory of Group Representations and Applications", PWN, Warszawa 1977.
3. I. I. BENEDIKTOVICH AND A. E. ZALESSKIJ, T-Ideals of free Lie algebras with polynomial growth of a sequence of codimensions, *Izv. Akad. Nauk BSSR Ser. Fiz. Mat.* 3 (1980), 5–10 (Russian).
4. A. BERELE, Homogeneous polynomial identities, *Israel J. Math.*, in press.

5. V. S. DRENSKY, Codimensions of T-ideals and Hilbert series of relatively free algebras. *C. R. Acad. Bulgare Sci.* **34** (1981). 1201–1204.
6. V. S. DRENSKY, On the lattices of varieties of associative algebras. *Serdica*, in press (Russian).
7. V. S. DRENSKY, Representations of the symmetric group and varieties of linear algebras. *Mat. Sb.* **115** (1981), 98–115 (Russian).
8. E. FORMANEK. The polynomial identities of matrices. Pennsylvania State University Preprint No. 8190, 1981.
9. E. FORMANEK. P. HALPIN. AND W.-C. W. LI. The Poincaré series of the ring of 2×2 generic matrices, *J. Algebra* **69** (1981). 105–112.
10. A. KERBER, "Representations of Permutation Groups." Lecture Notes in Matematics No 240, Springer-Verlag. Berlin/Heidelberg/New York. 1971.
11. D. KRAKOWSKY AND A. REGEV, The polynomial identities of the Grassman algebra. *Trans. Amer. Math. Soc.* **181** (1973), 429–438.
12. C. PROCESI. "Rings with Polynomial Identities." Dekker. New York, 1973.
13. JU. P. RAZMYSLOV. Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero, *Algebra i Logika* **12** (1973), 83–113 (Russian). Transl.: *Algebra Logic* **12** (1973), 47–63.
14. JU. P. RAZMYSLOV, On the Jacobson radical in PI-algebras, *Algebra i Logika* **13** (1974), 337–360 (Russian). Transl. *Algebra Logic* **13** (1974), 192–204.
15. A. REGEV. Existence of identities in $A \otimes B$. *Israel J. Math.* **11** (1972). 131–152.
16. A. REGEV, The polynomial identities of matrices in characteristic zero. *Comm. Algebra* **8** (1980). 1417–1467.
17. A. N. STOYANOVA-VENKOVA, The lattice of the variety of associative algebras defined by a commutator of length five. *C. R. Acad. Bulgare Sci.* **34** (1981). 465–468 (Russian).
18. H. WEYL. "The Classical Groups, Their Invariants and Representaions." Princeton Univ. Press. Princeton. N. J.. 1946.
19. I. B. VOLICHENKO, The T-ideal generated by the element $[X_1, X_2, X_3, X_4]$. Preprint No 22. Institut Matemat. Akad. Nauk BSSR, Minsk. 1978 (Russian).
20. K. A. ZHEVLAKOV, A. M. SLIN'KO. I. P. SHESTAKOV. AND A. I. SHIRSHOV, "Nearly Associative Rings," Sovremennaya Algebra. Nauka, Moskow. 1978 (Russian).